

# REFLECTING RANDOM FLIGHTS

ALESSANDRO DE GREGORIO AND ENZO ORSINGER

**ABSTRACT.** We consider random flights in  $\mathbb{R}^d$  reflecting on the surface of a sphere  $\mathbb{S}_R^{d-1}$ , with center at the origin and with radius  $R$ , where reflection is performed by means of circular inversion. Random flights studied in this paper are motions where the orientation of the deviations are uniformly distributed on the unit-radius sphere  $\mathbb{S}_1^{d-1}$ .

We obtain the explicit probability distributions of the position of the moving particle when the number of changes of direction is fixed and equal to  $n \geq 1$ . We show that these distributions involve functions which are solutions of the Euler-Darboux-Poisson equation. The unconditional probability distributions of the reflecting random flights are obtained by suitably randomizing  $n$  by means of a fractional-type Poisson process.

Random flights reflecting on hyperplanes according to the optical reflection form are considered and the related distributional properties derived.

## 1. INTRODUCTION

This paper is concerned with random flights, that is with random motions in the Euclidean space  $\mathbb{R}^d$ ,  $d \geq 2$ , performed at finite velocity (sometimes also called random evolutions). These continuous-time non-Markovian random motions have sample paths formed by straight lines, turning through any angle whatever, i.e. with uniformly distributed directions on the unit-radius sphere.

These random flights are useful to describe concrete motions of particles in gases and for these reason have been investigated by many physicists and mathematicians over the years. The first ones who analyzed the random flights with a fixed number of deviations were K. Pearson and J.C. Kluyver and some years later, S. Chandrasekar wrote a long paper on this subject with applications to astronomy.

Recent applications to the analysis of photon propagation in the Cosmic Microwave Background (CMB) radiation have been discussed in Reimberg and Abramo (2013). Furthermore, Martens *et al.* (2012) have shown that the probability law of planar random motions coincides with the explicit form of the van Hove function for the run-and-tumble model in two dimensions. This work gives an interesting and strong link between explicit solutions of the Lorentz model of electron conduction and the probability theory of random flights. For other possible applications of the random flights models, see Hughes (1995).

The main object of our investigation is represented by the position  $\{\mathbf{X}_d(t), t > 0\}$  reached after a fixed or a random number of deviations. The randomization of the number  $\mathcal{N}(t)$  of steps and of the lengths of intermediate displacements has produced more flexible versions of random flights for which explicit distributions of  $\{\mathbf{X}_d(t), t > 0\}$  have been obtained.

Recently many papers have studied random motions with velocity  $c > 0$ , with uniformly distributed deviations at Poisson paced times in the plane and successively in the Euclidean space  $\mathbb{R}^d$  (see, for instance, Stadje, 1987, Masoliver *et al.*, 1993, Kolesnik and Orsingher, 2005,

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for planar motions, Stadje, 1989, and Orsingher and De Gregorio, 2007, in  $\mathbb{R}^d$ ). Franceschetti, (2007) established a relationship between the number of deviations  $n$  and the dimension  $d$  for which the conditional distributions  $P\{\mathbf{X}_d(t) \in d\mathbf{x}_d | \mathcal{N}(t) = n\}$  are uniform. The assumption that changes of direction are governed by a homogeneous Poisson process leads to explicit (conditional and unconditional) probability distributions of  $\{\mathbf{X}_d(t), t > 0\}$  only for  $d = 2, 4$ . The idea of assuming different forms for intermediate steps (displacements) has produced fruitful results in that the probability distribution of  $\mathbf{X}_d(t)$  can be explicitly produced for all spaces of dimension  $d \geq 2$ .

The random flight  $(\underline{\Theta}, \underline{\tau}, \mathcal{N}(t))$ , is a triple where  $\underline{\Theta}$  represents the ensemble of deviations during the time interval  $[0, t]$ ,  $\underline{\tau}$  is the vector of the lengths of the displacements for a fixed number  $\mathcal{N}(t)$  number of changes of direction. These processes have been extensively investigated in the case where  $\underline{\tau}$  has a Dirichlet distribution and  $\underline{\Theta}$  is spherically uniform. The papers by Le Caër (2010), (2011), De Gregorio and Orsingher (2012), De Gregorio (2014) and Letac and Piccioni (2014) are devoted to this case. Pogorui and Rodriguez-Dagnino (2011), (2013) have studied random flights where  $\underline{\tau}$  has Erlang distribution, while Beghin and Orsingher (2010) obtained conditional distributions of  $\{\mathbf{X}_d(t), t > 0\}$  where the steps  $\underline{\tau}$  are Gamma random variables with parameter 2.

The equations governing the unconditional probability distributions of the random flight  $(\underline{\Theta}, \underline{\tau}, \mathcal{N}(t))$  have recently been obtained (see Garra and Orsingher, 2014). The case where  $\underline{\Theta}$  has a specific non-uniform distribution has been studied by De Gregorio (2012). Motions on subspaces of  $\mathbb{R}^d$  can be regarded as random flights with random velocities and are studied in De Gregorio and Orsingher (2012) and Pogorui and Rodriguez-Dagnino (2012). Asymptotic results for the position of these type of random walks have been obtained by Ghosh *et al.* (2014).

We now describe in detail the structure of the random flights. Let us consider a particle or a walker which starts from the origin of  $\mathbb{R}^d$ ,  $d \geq 2$ , and performs its motion with a constant velocity  $c > 0$ . We indicate by  $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots$  the random instants at which the random walker changes direction and denote the length of time separating these instants by  $\tau_k = t_k - t_{k-1}$ ,  $k \geq 1$ . Let  $\mathcal{N}(t) = \sup\{k \geq 1 : t_k \leq t\}$  be the (random) number of times in which the random motion changes direction during the interval  $[0, t]$ . If, at time  $t > 0$ ,  $\mathcal{N}(t) = n$ , with  $n \geq 1$ , the random motion has performed  $n + 1$  displacements. We observe that  $\underline{\tau}_n = (\tau_1, \dots, \tau_n) \in S_n$ , where  $S_n$  represents the open simplex

$$S_n = \left\{ (\tau_1, \dots, \tau_n) \in \mathbb{R}^n : 0 < \tau_k < t - \sum_{j=0}^{k-1} \tau_j, k = 1, 2, \dots, n \right\},$$

with  $\tau_0 = 0$  and  $\tau_{n+1} = t - \sum_{j=1}^n \tau_j$ .

By  $\underline{\Theta}_{n+1} = (\underline{\theta}_{d-1}^1, \dots, \underline{\theta}_{d-1}^k, \dots, \underline{\theta}_{d-1}^{n+1})$  we denote the vector of independent deviations performed at times  $t_k$ ,  $k = 0, 1, \dots, n$ . The  $k$ -th random variable  $\underline{\theta}_{d-1}^k$  is a  $(d-1)$ -dimensional random variable with components  $(\theta_1^k, \dots, \theta_{d-2}^k, \phi^k)$  distributed uniformly on the unit-radius sphere  $\mathbb{S}_1^{d-1} = \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d\| = 1\}$ . This means that the probability density function of  $\underline{\theta}_{d-1}^k$  is equal to

$$(1.1) \quad \varphi(\underline{\theta}_{d-1}^k) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \sin^{d-2} \theta_1^k \sin^{d-3} \theta_2^k \dots \sin \theta_{d-2}^k,$$

where  $\theta_j^k \in [0, \pi]$ ,  $j \in \{1, \dots, d-2\}$ ,  $\phi^k \in [0, 2\pi]$ . Furthermore,  $\tau_k$  and  $\theta_j^k$  are independent for each  $k$ .

Let us denote by  $\{\mathbf{X}_d(t), t > 0\}$  the process representing the position reached, at time  $t > 0$ , by the particle moving randomly according to the rules described above. The position  $\mathbf{X}_d(t) = (X_1(t), \dots, X_d(t))$ , is the main object of interest of the random flight and can be written, for

$\mathcal{N}(t) = n$ , as

$$(1.2) \quad \mathbf{X}_d(t) = c \sum_{k=1}^{n+1} \mathbf{V}_d^k \tau_k,$$

where  $\mathbf{V}_d^k, k = 1, 2, \dots, n+1$ , are independent  $d$ -dimensional random vectors defined as follows

$$\mathbf{V}_d^k = \begin{pmatrix} \sin \theta_1^k \sin \theta_2^k \cdots \sin \theta_{d-2}^k \sin \phi^k \\ \sin \theta_1^k \sin \theta_2^k \cdots \sin \theta_{d-2}^k \cos \phi^k \\ \dots \\ \sin \theta_1^k \cos \theta_2^k \\ \cos \theta_1^k \end{pmatrix}$$

and  $(\theta_1^k, \theta_2^k, \dots, \theta_{d-2}^k, \phi^k)$  are independent and identically distributed with density (1.1).

In this work we analyze reflecting random walks defined by means of inversive geometry. In particular, this paper studies random flights reflecting on the  $d$ -dimensional sphere  $\mathbb{S}_R^{d-1} = \{\mathbf{x}_d : \|\mathbf{x}_d\| = R\}$  with radius  $R$ . The reflected processes are constructed by suitably manipulating the sample paths of the free random flight  $\{\mathbf{X}_d(t), t > 0\}$ . We assume that the part of the trajectories of the free random flight are reflected by inversion with respect to the sphere  $\mathbb{S}_R^{d-1}$ . This produces substantial changes in their form, because the segments of outside lying sample paths are converted into arcs of circle inside  $\mathbb{S}_R^{d-1}$ . The picture of the reflecting processes is therefore made up by straight lines (for sample paths which never crossed  $\mathbb{S}_R^{d-1}$ ) and circular arcs (for the sample paths which performed excursions outside  $\mathbb{S}_R^{d-1}$ ). The same approach is used by Aryasova *et al.* (2013) in the study of reflecting diffusion processes.

The circular inversion permits us to write down the probability distribution  $p_n^*(\mathbf{x}_d, t)$  of the reflected process  $\{\mathbf{X}_d^*(t), t > 0\}$  by exploiting the density function  $p_n(\mathbf{x}_d, t)$  of the free random flight  $\{\mathbf{X}_d(t), t > 0\}$ . We show that

$$(1.3) \quad p_n^*(\mathbf{x}_d, t) = \begin{cases} p_n(\mathbf{x}_d, t) \mathbf{1}_{B_R^d}(\mathbf{x}_d), & t \leq \frac{R}{c} \\ p_n(\mathbf{x}_d, t) \mathbf{1}_{B_R^d}(\mathbf{x}_d) + \frac{R^{2d}}{\|\mathbf{x}_d\|^{2d}} p_n\left(R^2 \frac{\mathbf{x}_d}{\|\mathbf{x}_d\|^2}, t\right) \mathbf{1}_{C_{\frac{R^2}{ct}, R}^d}(\mathbf{x}_d), & t > \frac{R}{c}, \end{cases}$$

where  $B_R^d$  is the ball in the space  $\mathbb{R}^d$  with radius  $R$ ,  $C_{\frac{R^2}{ct}, R}^d = \{\mathbf{x}_d \in \mathbb{R}^d : \frac{R^2}{ct} < \|\mathbf{x}_d\| \leq R\}$  and  $\mathbf{1}_A(x)$  is the indicator function for the set  $A$ . For  $t > \frac{R}{c}$ , formula (1.3) registers the contribution of excursions outside  $\mathbb{S}_R^{d-1}$  which are brought inside the sphere  $\mathbb{S}_R^{d-1}$  by circular inversion. If  $p_n(\mathbf{x}_d, t)$  refers to random flights with Dirichlet distributed displacements we have that (see formulas (2.10) and (2.11) in De Gregorio and Orsingher, 2012)

$$(1.4) \quad p_n(\mathbf{x}_d, t) = \frac{\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{n}{2}(d-h))} (c^2 t^2 - \|\mathbf{x}_d\|^2)^{\frac{n}{2}(d-h)-1}, \quad \|\mathbf{x}_d\| < ct, h = 1, 2.$$

The space-dependent factor appearing in (1.4) is a function satisfying the Euler-Poisson-Darboux (EPD) equation. In other words, the function

$$f_\beta(\mathbf{x}_d, t) := (c^2 t^2 - \|\mathbf{x}_d\|^2)^\beta, \quad \beta \in \mathbb{R}, \|\mathbf{x}_d\| < ct,$$

satisfies the EPD equation

$$(1.5) \quad \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u + \frac{2\beta - 1 + d}{t} \frac{\partial u}{\partial t},$$

where  $\Delta := \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ , which becomes the  $d$ -dimensional wave equation for  $\beta = \frac{d-1}{2}$ . In the reflected motions considered here we must examine the functions

$$\bar{f}_\beta(\mathbf{x}_d, t) := \left( c^2 t^2 - \frac{R^4}{\|\mathbf{x}_d\|^2} \right)^\beta, \quad \beta \in \mathbb{R}, \frac{R^2}{ct} < \|\mathbf{x}_d\| < R,$$

which also are solutions of more complicated Euler-Poisson-Darboux equations where space-varying coefficients appear.

The final section of the paper is concerned with random flights reflecting when colliding with hyperplanes. In this case reflection is intended in the sense that striking and reflecting paths form the same angle with respect to the normal to the hyperplane  $H(\underline{a}_d, b) = \{\mathbf{x}_d \in \mathbb{R}^d : \langle \underline{a}_d, \mathbf{x}_d \rangle = b; \underline{a}_d \in \mathbb{R}^d, b \in \mathbb{R}\}$ . In this sense reflecting random flights behave as light rays of optics when the reflecting surface is a hyperplane. The trajectories of the reflecting random motions can be obtained from those of the free random flights by means of the bijective operator  $\nu : \mathbb{R}^d \rightarrow \mathbb{R}^d$  defined as

$$(1.6) \quad \nu(\mathbf{x}_d) := \mathbf{x}_d + 2 \frac{b - \langle \underline{a}_d, \mathbf{x}_d \rangle}{\langle \underline{a}_d, \underline{a}_d \rangle} \underline{a}_d,$$

which represents the reflection on the hyperplane  $H(\underline{a}_d, b)$ . For  $t > t' := \inf\{t : H(\underline{a}_d, b) \cap B_{ct}^d \neq \emptyset\}$ , we therefore have that the density  $p'_n(\mathbf{x}_d, t)$  of the reflected random motion  $\{\mathbf{X}'_d(t), t > 0\}$  reads

$$(1.7) \quad p'_n(\mathbf{x}_d, t) = \begin{cases} p_n(\mathbf{x}_d, t) \mathbf{1}_{B_{ct}^d}(\mathbf{x}_d), & t \leq t' \\ p_n(\mathbf{x}_d, t) \mathbf{1}_{L_{ct}^d}(\mathbf{x}_d) + p_n(\nu(\mathbf{x}_d), t) \mathbf{1}_{V_{ct}^d}(\mathbf{x}_d), & t > t', \end{cases}$$

where  $L_{ct}^d := L_{ct}^d(\underline{a}_d, b) := \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d\|^2 < c^2 t^2, \langle \underline{a}_d, \mathbf{x}_d \rangle < b\}$  and  $V_{ct}^d$  consists of points of  $B_{ct}^d$  reflected by the operator  $\nu$  (see Figure 2).

The main concepts of the inversive geometry represent fundamental tools for our analysis. Therefore, for the convenience of the reader, we recall some basic definitions in the Appendices A-B.

## 2. NOTATIONS

We list the main symbols used in this paper.

- With  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  we indicate the Euclidean norm and the scalar product, respectively.
- Let  $\mathbb{S}_R^{d-1}(\mathbf{x}_d^0) := \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d - \mathbf{x}_d^0\| = R\}$  be the sphere with radius  $R > 0$  and center at  $\mathbf{x}_d^0 \in \mathbb{R}^d$ . We set  $\mathbb{S}_R^{d-1}(O) := \mathbb{S}_R^{d-1}$  where  $O$  is the origin of  $\mathbb{R}^d$ . Furthermore, the surface area of  $\mathbb{S}_1^{d-1}$  is given by  $\text{area}(\mathbb{S}_1^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ .
- Let  $B_R^d := \{\mathbf{x}_d \in \mathbb{R}^d : \|\mathbf{x}_d\| < R\}$  be the ball with center  $O$  and radius  $R$ . Let  $C_{R_1, R_2}^d := \{\mathbf{x}_d \in \mathbb{R}^d : R_1 < \|\mathbf{x}_d\| \leq R_2\}$ .
- Let  $H(\underline{a}_d, b) := \{\mathbf{x}_d \in \mathbb{R}^d : \langle \underline{a}_d, \mathbf{x}_d \rangle = b; \underline{a}_d \in \mathbb{R}^d, b \in \mathbb{R}\}$  be a hyperplane in  $\mathbb{R}^d$ .
- Let  $\mathbf{1}_A(x)$  be the indicator function, that is

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

while

$$J_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{2k+\nu} \frac{1}{k! \Gamma(k+\nu+1)}, \quad x, \nu \in \mathbb{R},$$

is the Bessel function of order  $\nu$ .

- Let  $n$  be the fixed number of changes of direction in the time interval  $[0, t]$ , we indicate the conditional probability areaure by

$$P_n\{\cdot \in A\} := P\{\cdot \in A | \mathcal{N}(t) = n\},$$

for all Borel sets  $A$ , with  $n \geq 1$ . Furthermore, let  $E_n\{\cdot\} := E\{\cdot | \mathcal{N}(t) = n\}$ .

- Let

$$\begin{aligned} p_n(\underline{\mathbf{x}}_d, t) &:= P_n\{\underline{\mathbf{X}}_d(t) \in d\underline{\mathbf{x}}_d\} / \prod_{k=1}^d dx_k, \\ p_n^*(\underline{\mathbf{x}}_d, t) &:= P_n\{\underline{\mathbf{X}}_d^*(t) \in d\underline{\mathbf{x}}_d\} / \prod_{k=1}^d dx_k, \\ p_n'(\underline{\mathbf{x}}_d, t) &:= P_n\{\underline{\mathbf{X}}_d'(t) \in d\underline{\mathbf{x}}_d\} / \prod_{k=1}^d dx_k. \end{aligned}$$

### 3. PRELIMINARY RESULTS ON RANDOM FLIGHTS

Let us indicate by  $g(\underline{\mathcal{I}}_n; t)$  the probability density function of the random vector  $\underline{\mathcal{I}}_n$ . We provide the probability distribution of  $\{\underline{\mathbf{X}}_d(t), t > 0\}$  when the number of steps performed by the motion in  $[0, t]$  is fixed.

**Lemma 1.** *Let  $n \geq 1$  be the number of changes of direction happening during time interval  $[0, t]$ . The conditional density function of  $\{\underline{\mathbf{X}}_d(t), t > 0\}$  is equal to*

$$(3.1) \quad p_n(\underline{\mathbf{x}}_d, t) = \frac{\left\{2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right)\right\}^{n+1}}{(2\pi)^{\frac{d}{2}} \|\underline{\mathbf{x}}_d\|^{\frac{d}{2}-1}} \int_0^\infty \left[ \int_{S_n} g(\underline{\mathcal{I}}_n; t) \prod_{k=1}^{n+1} \left\{ \frac{J_{\frac{d}{2}-1}(c\tau_k \rho)}{(c\tau_k \rho)^{\frac{d}{2}-1}} \right\} \prod_{k=1}^n d\tau_k \right] \rho^{\frac{d}{2}} J_{\frac{d}{2}-1}(\rho \|\underline{\mathbf{x}}_d\|) d\rho,$$

with  $\|\underline{\mathbf{x}}_d\| < ct$ .

*Proof.* The expression (3.1) has been obtained by Orsingher and De Gregorio (2007) (formula (2.13)) in the uniform case and by De Gregorio (2014) (formula (3.1)) in the case of generalized Dirichlet distributions. Similar steps can be used in a general framework for the distribution  $g(\underline{\mathcal{I}}_n; t)$ . In what follows, we will provide a sketch of the proof.

Let us start the proof by showing that the characteristic function of  $\underline{\mathbf{X}}_d(t)$  is equal to

$$(3.2) \quad \begin{aligned} \mathcal{F}_n(\underline{\alpha}_d) &:= E_n \left\{ e^{i\langle \underline{\alpha}_d, \underline{\mathbf{X}}_d(t) \rangle} \right\} \\ &= \left\{ 2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right) \right\}^{n+1} \int_{S_n} g(\underline{\mathcal{I}}_n; t) \prod_{k=1}^{n+1} \left\{ \frac{J_{\frac{d}{2}-1}(c\tau_k \|\underline{\alpha}_d\|)}{(c\tau_k \|\underline{\alpha}_d\|)^{\frac{d}{2}-1}} \right\} \prod_{k=1}^n d\tau_k. \end{aligned}$$

We can write that

$$\mathcal{F}_n(\underline{\alpha}_d) = \int_{S_n} g(\underline{\mathcal{I}}_n; t) \mathcal{I}_n(\underline{\alpha}_d; \underline{\mathcal{I}}_d) \prod_{k=1}^n d\tau_k$$

where

$$(3.3) \quad \mathcal{I}_n(\underline{\alpha}_d; \underline{\mathcal{I}}_d) := \prod_{k=1}^{n+1} \left\{ \int_{\Lambda} \exp\{ic\tau_k \langle \underline{\alpha}_d, \underline{\mathbf{V}}_k \rangle\} \varphi(\underline{\theta}_{d-1}^k) \prod_{j=1}^{d-2} d\theta_j^k d\phi^k \right\}$$

where  $\Lambda := [0, \pi]^n \times [0, 2\pi]$ .

It is well-known that the integral  $\mathcal{I}_n(\underline{\alpha}_d; \underline{\tau}_d)$  (see Theorem 2.1 of Orsingher and De Gregorio, 2007, and formula (2.5) of De Gregorio and Orsingher, 2012) is equal to

$$(3.4) \quad \mathcal{I}_n(\underline{\alpha}_d; \underline{\tau}_d) = \left\{ 2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right) \right\}^{n+1} \prod_{k=1}^{n+1} \frac{J_{\frac{d}{2}-1}(c\tau_k \|\underline{\alpha}_d\|)}{(c\tau_k \|\underline{\alpha}_d\|)^{\frac{d}{2}-1}}$$

and this leads to (3.2).

Now, by inverting the characteristic function (3.2), we are able to show that the density of the process  $\{\underline{\mathbf{X}}_d(t), t > 0\}$ , is given by (3.1). Let us denote by  $\underline{v}_d$  the vector

$$\underline{v}_d = \begin{pmatrix} \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \sin \phi \\ \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{d-2} \cos \phi \\ \dots \\ \sin \theta_1 \cos \theta_2 \\ \cos \theta_1 \end{pmatrix}.$$

Therefore, by inverting the characteristic function (3.2) and by passing to  $d$ -dimensional spherical coordinates, we have, for  $\underline{\mathbf{x}}_d \in B_{ct}^d$ , that

$$\begin{aligned} p_n(\underline{\mathbf{x}}_d, t) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \underline{\alpha}_d, \underline{\mathbf{x}}_d \rangle} \mathcal{F}_n(\underline{\alpha}_d) \prod_{k=1}^d d\alpha_k \\ &= \frac{1}{(2\pi)^d} \int_0^\infty \rho^{d-1} d\rho \int_{\Lambda} e^{-i\rho \langle \underline{v}_d, \underline{\mathbf{x}}_d \rangle} d\mathbb{S}_1^{d-1} \left\{ 2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right) \right\}^{n+1} \int_{S_n} g(\underline{\tau}_n; t) \prod_{k=1}^{n+1} \left\{ \frac{J_{\frac{d}{2}-1}(c\tau_k \rho)}{(c\tau_k \rho)^{\frac{d}{2}-1}} \right\} \prod_{k=1}^n d\tau_k. \end{aligned}$$

where  $d\mathbb{S}_1^{d-1} := \prod_{i=1}^{d-2} \{\sin^{d-i-1} \theta_i d\theta_i\} d\phi$ . By applying formula (2.15) of Orsingher and De Gregorio (2007)

$$(3.5) \quad \int_{\Lambda} e^{-i\rho \langle \underline{v}_d, \underline{\mathbf{x}}_d \rangle} d\mathbb{S}_1^{d-1} = (2\pi)^{\frac{d}{2}} \frac{J_{\frac{d}{2}-1}(\rho \|\underline{\mathbf{x}}_d\|)}{(\rho \|\underline{\mathbf{x}}_d\|)^{\frac{d}{2}-1}},$$

we arrive at the claimed result.  $\square$

In the analysis of random flights, a central role is played by the probabilistic assumptions on the displacements  $c\underline{\tau}_n$ . It is useful to assume that the random vector  $\underline{\tau}_n$  has the following Dirichlet distribution

$$(3.6) \quad g(\underline{\tau}_n; \underline{\mathbf{q}}_{n+1}, t) = \frac{\Gamma((n+1)(\frac{d}{h}-1))}{(\Gamma(\frac{d}{h}-1))^{n+1}} \frac{1}{t^{(n+1)(\frac{d}{h}-1)}} \prod_{k=1}^{n+1} \tau_k^{\frac{d}{h}-2},$$

where  $\underline{\tau}_n \in S_n$ , with parameters  $\underline{\mathbf{q}}_{n+1} := (\frac{d}{h}-1, \dots, \frac{d}{h}-1)$ ,  $h := 1, 2$ , and the conditions  $d \geq 2$  if  $h = 1$  and  $d \geq 3$  if  $h = 2$  hold. Under the assumptions (3.6), the density function (3.1) can be evaluated explicitly

$$(3.7) \quad p_{n,h}(\underline{\mathbf{x}}_d, t) = \frac{\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{n}{2}(d-h))} \frac{(c^2 t^2 - \|\underline{\mathbf{x}}_d\|)^{\frac{n}{2}(d-h)-1}}{\pi^{\frac{d}{2}} (ct)^{(n+1)(d-h)+h-2}} \mathbf{1}_{B_R^d}(\underline{\mathbf{x}}_d),$$

while for the process  $\{D_d(t) = \|\underline{\mathbf{X}}_d(t)\|, t > 0\}$ , we are able to obtain the following conditional density

$$(3.8) \quad q_{n,d,h}(r, t) = \frac{2\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{n}{2}(d-h))} \frac{r^{d-1} (c^2 t^2 - r^2)^{\frac{n}{2}(d-h)-1}}{(ct)^{(n+1)(d-h)+h-2}} \mathbf{1}_{(0,R)}(r),$$

(see De Gregorio and Orsingher (2012) and Le Caër (2010)). We observe that  $\{D_d(t), t > 0\}$ , is related to the Beta distribution. Indeed,  $D_d(t)/c^2 t^2$  is distributed as a Beta random variable with parameters  $\frac{d}{2}$  and  $\frac{n}{2}(d-h)$ .

#### 4. REFLECTING RANDOM FLIGHTS IN SPHERES

**4.1. Definition and probability distributions.** Let us consider a random flight  $\{\mathbf{X}_d(t), t > 0\}$ , defined as in the Introduction. When a sample path of  $\mathbf{X}_d(t)$  strikes the sphere  $\mathbb{S}_R^{d-1}$ , the trajectory of the process is reflected inside  $\mathbb{S}_R^{d-1}$ . The reflection of the random flight on the boundary  $\mathbb{S}_R^{d-1}$  can be envisaged in different ways. The specular reflection (the incoming sample path forms the same angle as the reflected trajectory with respect to the normal vector to  $\mathbb{S}_R^{d-1}$ ) seems the most natural one. Nevertheless, from the mathematical point of view, the reflecting surface is closed and this implies that the probability distribution of the reflected process takes a cumbersome form for sufficiently large values of  $t$ . For this reason, we assume that the reflection is based on the principle of circular inversion in spheres defined in Appendix A. The most important effect of this procedure is that the sample paths obtained by reflection are deformed and take the structure of circumference arcs. This leads to a new process, namely, the reflecting random flight moving in  $B_R^d \cup \mathbb{S}_R^{d-1}$ .

**Definition 1.** *The reflecting random flight  $\{\mathbf{X}_d^*(t), t > 0\}$ , reflected on the sphere  $\mathbb{S}_R^{d-1}$  is constructed by means of the free process  $\{\mathbf{X}_d(t), t > 0\}$  as follows: 1) if  $t \leq \frac{R}{c}$ , then  $\mathbf{X}_d^*(t) := \mathbf{X}_d(t)$ ; 2) if  $t > \frac{R}{c}$ , then if at least one change of direction happens during the time interval  $[0, t]$ , we have that*

$$(4.1) \quad \mathbf{X}_d^*(t) := \mathbf{X}_d(t) \mathbf{1}_{B_R^d}(\mathbf{X}_d(t)) + \mu_R(\mathbf{X}_d(t)) \mathbf{1}_{C_{R,ct}^d}(\mathbf{X}_d(t)),$$

where  $\mu_R(\mathbf{x}_d) = R^2 \frac{\mathbf{x}_d}{\|\mathbf{x}_d\|^2}$  is the inversion map defined by (A.1); while if there are no deviations

$$(4.2) \quad \mathbf{X}_d^*(t) := \mu_R(\mathbf{X}_d(t)) \mathbf{1}_{\mathbb{S}_R^{d-1}}(\mathbf{X}_d(t)).$$

Definition 1 leads to the following considerations on the sample paths of  $\{\mathbf{X}_d^*(t), t > 0\}$ :

- the reflecting random flight has two components: the first one is given by the free process  $\{\mathbf{X}_d(t), t > 0\}$ ; the second component  $\mu_R(\mathbf{X}_d(t))$  is due to the reflection in  $\mathbb{S}_R^{d-1}$  of the sample paths of  $\{\mathbf{X}_d(t), t > 0\}$ , wandering outside the sphere  $\mathbb{S}_R^{d-1}$ . It is worth mentioning that the property P4) implies that the reflected paths have opposite orientation w.r.t. the trajectories of  $\{\mathbf{X}_d(t), t > 0\}$  moving outside  $\mathbb{S}_R^{d-1}$ ;
- the reflecting component of  $\{\mathbf{X}_d^*(t), t > 0\}$  is given by

$$(4.3) \quad \mu_R(\mathbf{X}_d(t)) = \frac{R^2}{\|\mathbf{X}_d(t)\|^2} \mathbf{X}_d(t) = \frac{cR^2}{\|\mathbf{X}_d(t)\|^2} \sum_{k=1}^{n+1} \mathbf{V}_k \tau_k,$$

for  $\mathbf{X}_d(t) \in C_{R,ct}^d$ . Therefore,  $\frac{cR^2}{\|\mathbf{X}_d(t)\|^2}$  can be thought of as the random velocity of (4.3).

Since  $\frac{cR^2}{\|\mathbf{X}_d(t)\|^2} \leq c$ , the reflected motion travels more slowly than the one related to the process  $\{\mathbf{X}_d(t), t > 0\}$ . For instance, see sample path **b** in Figure 1;

- the property P5) of the map  $\mu_R$  implies that the sample paths of  $\{\mathbf{X}_d^*(t), t > 0\}$  can be represented by broken lines, when no reflection has occurred, and by the composition of straight lines and circumference arcs, when at least one reflection has taken place (see sample path **a** in Figure 1).

We are able to provide the conditional probability distributions  $\{\mathbf{X}_d^*(t), t > 0\}$ , by means of those of  $\{\mathbf{X}_d(t), t > 0\}$ . Therefore, by means of (3.1), we are able to obtain the next result.

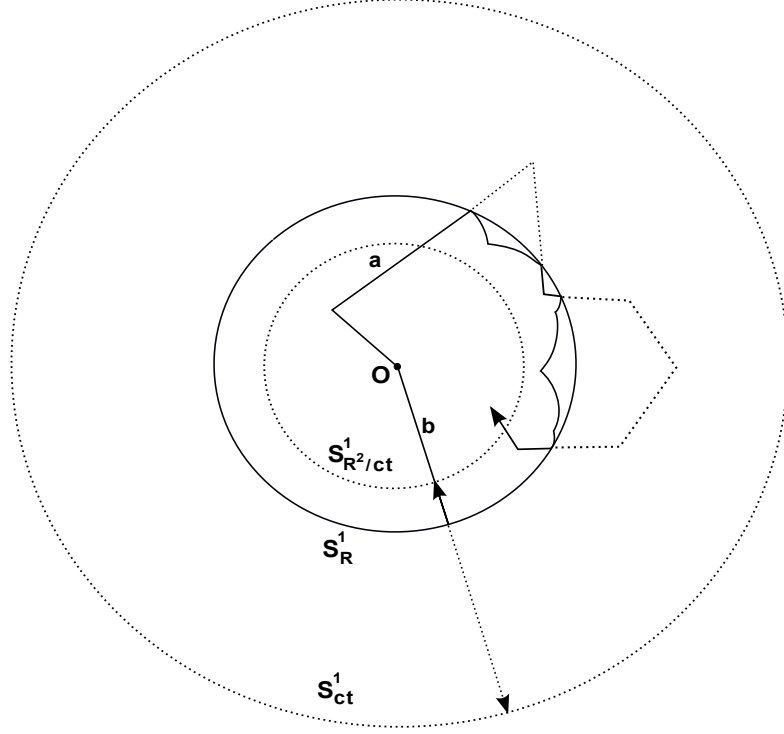


FIGURE 1. Two typical sample paths are depicted. The trajectory **a** is composed of straight lines and arc of spheres, while the sample path **b** is obtained if no changes of direction happen in the interval  $[0, t]$ .

**Theorem 1.** *If  $\mathcal{N}(t) = n$ , with  $n \geq 1$ , the process  $\{\underline{\mathbf{X}}_d^*(t), t > 0\}$ , has the following conditional density function*

$$(4.4) \quad p_n^*(\underline{\mathbf{x}}_d, t) = \begin{cases} p_n(\underline{\mathbf{x}}_d, t) \mathbf{1}_{B_R^d}(\underline{\mathbf{x}}_d), & t \leq \frac{R}{c} \\ p_n(\underline{\mathbf{x}}_d, t) \mathbf{1}_{B_R^d}(\underline{\mathbf{x}}_d) + \frac{R^{2d}}{\|\underline{\mathbf{x}}_d\|^{2d}} p_n\left(R^2 \frac{\underline{\mathbf{x}}_d}{\|\underline{\mathbf{x}}_d\|^2}, t\right) \mathbf{1}_{C_{\frac{R^2}{ct}, R}^d}(\underline{\mathbf{x}}_d), & t > \frac{R}{c}. \end{cases}$$

where  $p_n(\underline{\mathbf{x}}_d, t)$  is equal to (3.1).

*Proof.* The case  $t \leq \frac{R}{c}$  is obvious. Now we assume that  $t > \frac{R}{c}$ . Let  $A$  be a Borel set such that  $A \cap \overline{B_R^d} \neq \emptyset$ . Let  $\underline{\mathbf{y}}_d(t) := \mu_R(\underline{\mathbf{X}}_d(t)) = R^2 \frac{\underline{\mathbf{X}}_d(t)}{\|\underline{\mathbf{X}}_d(t)\|^2}$ , we get

$$(4.5) \quad \begin{aligned} P_n\{\underline{\mathbf{X}}_d^*(t) \in A\} &= P_n\{\underline{\mathbf{X}}_d(t) \in A \cap B_R^d\} + P_n\left\{\underline{\mathbf{y}}_d(t) \in A \cap C_{\frac{R^2}{ct}, R}^d\right\} \\ &= \int_{A \cap B_R^d} p_n(\underline{\mathbf{x}}_d, t) \prod_{k=1}^d dx_k + P_n\left\{\underline{\mathbf{y}}_d(t) \in A \cap C_{\frac{R^2}{ct}, R}^d\right\}. \end{aligned}$$

The first term in the previous expression refers to the sample paths which do not attain the boundary  $\mathbb{S}_R^{d-1}$  up to time  $t$ , while the second probability concerns the trajectories reflecting on  $\mathbb{S}_R^{d-1}$ . By setting

$$D_A := \mu_R^{-1}\left(A \cap C_{\frac{R^2}{ct}, R}^d\right) = \left\{\underline{\mathbf{y}}_d \in \mathbb{R}^d : \underline{\mathbf{x}}_d = \mu_R(\underline{\mathbf{y}}_d) \in A \cap C_{\frac{R^2}{ct}, R}^d\right\}$$



then, by means of Jacobi transformation formula, we have that

$$\begin{aligned}
 P_n \left\{ \underline{\mathbf{Y}}_d(t) \in A \cap C_{\frac{R^2}{ct}, R}^d \right\} &= P_n \{ \underline{\mathbf{X}}_d(t) \in D_A \} \\
 &= \int_{D_A} p_n(\underline{\mathbf{y}}_d, t) \prod_{k=1}^d dy_k \\
 &= \int_{A \cap C_{\frac{R^2}{ct}, R}^d} p_n(\mu_R^{-1}(\underline{\mathbf{x}}_d), t) |\det(J_{\mu_R^{-1}}(\underline{\mathbf{x}}_d))| \prod_{k=1}^d dx_k \\
 (4.6) \quad &= \int_{A \cap C_{\frac{R^2}{ct}, R}^d} p_n(\mu_R(\underline{\mathbf{x}}_d), t) |\det(J_{\mu_R}(\underline{\mathbf{x}}_d))| \prod_{k=1}^d dx_k
 \end{aligned}$$

where in the last step we have used the following facts:  $\mu_R = \mu_R^{-1}$  (which follows from the property P3)) and  $|\det(J_{\mu_R}(\underline{\mathbf{x}}_d))| = \frac{R^{2d}}{\|\underline{\mathbf{x}}_d\|^{2d}}$ . From (4.5) and (4.6), the result (4.4) immediately follows.  $\square$

From Lemma 1 emerges that the random process  $\{\underline{\mathbf{X}}_d(t), t > 0\}$  is isotropic, namely  $Q(\underline{\mathbf{X}}_d(t)) \sim \underline{\mathbf{X}}_d(t)$  for all  $Q \in O(d)$  and  $p_n(\underline{\mathbf{x}}_d, t) = p_n(\|\underline{\mathbf{x}}_d\|, t)$ . Furthermore, from (3.1), we have that  $\{\underline{\mathbf{X}}_d^*(t), t > 0\}$  is invariant by rotation as well. Therefore,  $p_n^*(\underline{\mathbf{x}}_d, t)$  depends on the distance  $\|\underline{\mathbf{x}}_d\|$  and thus we can write

$$\begin{aligned}
 p_n^*(\underline{\mathbf{x}}_d, t) &= p_n^*(\|\underline{\mathbf{x}}_d\|, t) \\
 (4.7) \quad &= p_n(\|\underline{\mathbf{x}}_d\|, t) \mathbf{1}_{(0, R)}(\|\underline{\mathbf{x}}_d\|) + \frac{R^{2d}}{\|\underline{\mathbf{x}}_d\|^{2d}} p_n\left(\frac{R^2}{\|\underline{\mathbf{x}}_d\|}, t\right) \mathbf{1}_{(R^2/ct, R]}(\|\underline{\mathbf{x}}_d\|)
 \end{aligned}$$

for  $t > \frac{R}{c}$ .

**Definition 2.** The reflecting radial process  $\{D_d^*(t), t > 0\}$ , represents the Euclidean distance from  $\underline{\mathbf{0}}_d$  of the position  $\underline{\mathbf{X}}_d^*(t)$ , namely  $D_d^*(t) = \|\underline{\mathbf{X}}_d^*(t)\|$ . It can be defined by

$$(4.8) \quad D_d^*(t) = D_d(t) \mathbf{1}_{(0, R)}(D_d(t)) + \frac{R^2}{D_d(t)} \mathbf{1}_{[R, ct]}(D_d(t)),$$

where  $D_d(t) = \|\underline{\mathbf{X}}_d(t)\|$ .

As a consequence of the isotropic structure of  $\{\underline{\mathbf{X}}_d^*(t), t > 0\}$ , the conditional density function of  $\{D_d^*(t), t > 0\}$ , becomes

$$(4.9) \quad q_{n,d}^*(r, t) = r^{d-1} \text{area}(\mathbb{S}_1^{d-1}) p_n^*(r, t) \mathbf{1}_{(0, R]}(r).$$

**4.2. Reflecting Dirichlet random flights.** A suitable choice of the distribution  $g(\mathcal{I}_n; t)$  leads to explicit expressions for the conditional density functions (4.4). As we have seen in Section 3 the Dirichlet distributions play a special role in the study of random flights. The assumption (3.6) and the results (3.7) and (3.8) imply that the reflecting process  $\{\underline{\mathbf{X}}_d^*(t), t > 0\}$  has probability law (4.4) given by

$$\begin{aligned}
 (4.10) \quad p_{n,h}^*(\underline{\mathbf{x}}_d, t) &= \frac{\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{n}{2}(d-h))} \frac{1}{\pi^{\frac{d}{2}}(ct)^{(n+1)(d-h)+h-2}} \\
 &\times \left[ (c^2 t^2 - \|\underline{\mathbf{x}}_d\|^2)^{\frac{n}{2}(d-h)-1} \mathbf{1}_{B_R^d}(\underline{\mathbf{x}}_d) + \frac{R^{2d}}{\|\underline{\mathbf{x}}_d\|^{2d}} \left( c^2 t^2 - \frac{R^4}{\|\underline{\mathbf{x}}_d\|^2} \right)^{\frac{n}{2}(d-h)-1} \mathbf{1}_{C_{\frac{R^2}{ct}, R}^d}(\underline{\mathbf{x}}_d) \right],
 \end{aligned}$$

with  $t > \frac{R}{c}$ , while  $\{D_d^*(t), t > 0\}$  has the following conditional distribution

$$(4.11) \quad q_{n,d,h}^*(r, t) = \frac{2\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{n}{2}(d-h))(ct)^{(n+1)(d-h)+h-2}} \\ \times \left[ r^{d-1}(c^2t^2 - r^2)^{\frac{n}{2}(d-h)-1} \mathbf{1}_{(0,R)}(r) + \frac{R^{2d}}{r^{d+1}} \left( c^2t^2 - \frac{R^4}{r^2} \right)^{\frac{n}{2}(d-h)-1} \mathbf{1}_{(R^2/ct, R]}(r) \right],$$

with  $t > \frac{R}{c}$ . In this case we call these random walks “reflecting Dirichlet random flights”.

**Remark 4.1.** The probability  $P_n\{\mathbf{X}_d(t) \in d\mathbf{x}_d\}$  is uniform inside the ball  $B_{ct}^d$  in the following cases: (i)  $h = 1, d = 2, n = 2$ ; (ii)  $h = 1, d = 3, n = 1$ ; (iii)  $h = 2, d = 3, n = 2$ ; (iv)  $h = 2, d = 4, n = 1$ .

In the cases (i)-(iv), we have that the function  $p_{n,h}^*(\mathbf{x}_d, t)$  becomes

$$p_{2,1}^*(\mathbf{x}_2, t) = \frac{1}{\pi(ct)^2} \left[ \mathbf{1}_{B_R^2}(\mathbf{x}_2) + \frac{R^4}{\|\mathbf{x}_2\|^4} \mathbf{1}_{C_{\frac{R^2}{ct}, R}^2}(\mathbf{x}_2) \right], \\ p_{1,1}^*(\mathbf{x}_3, t) = p_{2,2}^*(\mathbf{x}_3, t) = \frac{\Gamma(\frac{5}{2})}{\pi^{\frac{3}{2}}(ct)^3} \left[ \mathbf{1}_{B_R^3}(\mathbf{x}_3) + \frac{R^6}{\|\mathbf{x}_3\|^6} \mathbf{1}_{C_{\frac{R^2}{ct}, R}^3}(\mathbf{x}_3) \right], \\ p_{1,2}^*(\mathbf{x}_4, t) = \frac{\Gamma(3)}{\pi^2(ct)^4} \left[ \mathbf{1}_{B_R^4}(\mathbf{x}_4) + \frac{R^8}{\|\mathbf{x}_4\|^8} \mathbf{1}_{C_{\frac{R^2}{ct}, R}^4}(\mathbf{x}_4) \right].$$

This implies that the reflecting random flights are never uniformly distributed inside the ball  $B_R^d$ .

Now, we focus our attention on the distribution function of  $\{D_d^*(t), t > 0\}$  which also provides information on the probability of the position of the reflecting random flight at time  $t > 0$ , that is

$$P_n\{D_d^*(t) < r\} = P_n\{\mathbf{X}_d^*(t) \in B_r^d\}, \quad 0 < r \leq R.$$

The probability distribution of the distance process  $\{D_d^*(t), t > 0\}$  is related to the Beta distribution as shown in the next result. Let  $r \in (0, R]$  and  $t > \frac{R}{c}$ , by using (3.8), we immediately obtain that

$$(4.12) \quad P_n\{D_d^*(t) < r\} = \int_0^{\frac{r^2}{(ct)^2}} \frac{x^{\frac{d}{2}-1}(1-x)^{\frac{n}{2}(d-h)-1}}{B\left(\frac{d}{2}, \frac{n}{2}(d-h)\right)} dx + \left[ \int_{\frac{R^4}{(ctr)^2}}^1 \frac{x^{\frac{d}{2}-1}(1-x)^{\frac{n}{2}(d-h)-1}}{B\left(\frac{d}{2}, \frac{n}{2}(d-h)\right)} dx \right] \mathbf{1}_{(R^2/ct, R]}(r) \\ = \frac{B\left(\frac{r^2}{(ct)^2}; \frac{d}{2}, \frac{n}{2}(d-h)\right)}{B\left(\frac{d}{2}, \frac{n}{2}(d-h)\right)} + \left[ 1 - \frac{B\left(\frac{R^4}{(ctr)^2}; \frac{d}{2}, \frac{n}{2}(d-h)\right)}{B\left(\frac{d}{2}, \frac{n}{2}(d-h)\right)} \right] \mathbf{1}_{(R^2/ct, R]}(r)$$

where  $B(a, b) := \Gamma(a)\Gamma(b)/\Gamma(a+b)$  and  $B(z; a, b) := \int_0^z u^{a-1}(1-u)^{b-1} du$  is the incomplete gamma function. From (4.12) we obtain that

$$P_n\{\mathbf{X}_d^*(t) \in C_{\frac{R^2}{ct}, R}^d\} = P_n\left\{ \frac{R^2}{ct} < D_d^*(t) \leq R \right\} \\ = 1 - B\left(R^4; \frac{d}{2}, \frac{n}{2}(d-h)\right)$$

The probability (4.12) becomes particularly simple for  $d = 2$  and  $h = 1$ . Indeed, we have that

$$(4.13) \quad P_n\{D_2^*(t) < r\} = 1 - \left(1 - \frac{r^2}{(ct)^2}\right)^{\frac{n}{2}} + \left(1 - \frac{R^4}{(ctr)^2}\right)^{\frac{n}{2}} \mathbf{1}_{(R^2/ct, R]}(r), \quad r \in (0, R].$$

Furthermore, if we assume that  $h = 2, d = 2d', d' \geq 2$ , it is possible to write down  $P_n\{D_d^*(t) < r\}$  by means of the probability distribution of binomial r.v.'s. By exploiting the following well-known result

$$(4.14) \quad \frac{B(x; a, b)}{B(a, b)} = \sum_{k=a}^{a+b-1} \binom{a+b-1}{k} x^k (1-x)^{a+b-1-k}, \quad a, b \in \mathbb{N},$$

it is not hard to prove that

$$(4.15) \quad P_n\{D_d^*(t) < r\} = P\left\{\frac{d}{2} \leq Y_{n,d} \leq \frac{n+1}{2}(d-2)\right\} + P\left\{0 \leq \hat{Y}_{n,d} < \frac{d}{2}\right\} \mathbf{1}_{(R^2/ct, R]}(r)$$

where  $Y_{n,d} \sim \text{Bin}\left(\frac{n+1}{2}(d-2), \frac{r^2}{(ct)^2}\right)$  and  $\hat{Y}_{n,d} \sim \text{Bin}\left(\frac{n+1}{2}(d-2), \frac{R^4}{(ctr)^2}\right)$ .

For  $t > \frac{R}{c}$ , we are also able to derive the  $m$ -th moment, with  $m \geq 1$ , of  $\{D_d^*(t), t > 0\}$ . We have that

$$(4.16) \quad E_n\{D_d^*(t)\}^m = \frac{1}{B\left(\frac{d}{2}, \frac{n}{2}(d-h)\right)} \left[ (ct)^m \int_0^{\frac{R^2}{(ct)^2}} x^{\frac{d+m}{2}-1} (1-x)^{\frac{n}{2}(d-h)-1} dx \right. \\ \left. + \left(\frac{R^2}{ct}\right)^m \int_{\frac{R^2}{(ct)^2}}^1 x^{\frac{d-m}{2}-1} (1-x)^{\frac{n}{2}(d-h)-1} dx \right].$$

Moreover, if  $d > m$ , we can write (4.16) in terms of beta and incomplete beta functions as follows

$$(4.17) \quad E_n\{D_d^*(t)\}^m = (ct)^m \frac{B\left(\frac{R^2}{(ct)^2}; \frac{d+m}{2}, \frac{n}{2}(d-h)\right)}{B\left(\frac{d}{2}, \frac{n}{2}(d-h)\right)} + \left(\frac{R^2}{ct}\right)^m \left[ 1 - \frac{B\left(\frac{R^2}{(ct)^2}; \frac{d-m}{2}, \frac{n}{2}(d-h)\right)}{B\left(\frac{d}{2}, \frac{n}{2}(d-h)\right)} \right].$$

**4.3. Random flights and the Euler-Poisson-Darboux equation.** In De Gregorio and Orsingher (2012) (Remark 2.7) is observed that the functions

$$(4.18) \quad f_\beta(\mathbf{x}_d, t) := (c^2 t^2 - \|\mathbf{x}_d\|^2)^\beta, \quad \|\mathbf{x}_d\| < ct, \beta \in \mathbb{R},$$

satisfy the following Euler-Poisson-Darboux (EPD) partial differential equation

$$(4.19) \quad \frac{\partial^2 u}{\partial t^2} - \frac{2\beta - 1 + d}{t} \frac{\partial u}{\partial t} = c^2 \Delta u$$

where  $\Delta := \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ . If  $\beta = \frac{n}{2}(d-h) - 1$ , we have that

$$f_{\frac{n}{2}(d-h)-1}(\mathbf{x}_d, t) = \frac{\pi^{\frac{d}{2}} (ct)^{(n+1)(d-h)+h-2} \Gamma\left(\frac{n}{2}(d-h)\right)}{\Gamma\left(\frac{n+1}{2}(d-h) + \frac{h}{2}\right)} p_{n,h}(\mathbf{x}_d, t).$$

Therefore, by exploiting the equation (4.19), it is not hard to show that  $p_{n,h}(\mathbf{x}_d, t)$  is a solution of the following EPD equation

$$(4.20) \quad \frac{\partial^2 u}{\partial t^2} + \frac{(n+1)(d-h) + h - 1}{t} \frac{\partial u}{\partial t} = c^2 \Delta u.$$

The projection of the random process  $\{\mathbf{X}_d(t), t > 0\}$  onto a lower space of dimension  $m$ , implies that the conditional marginal distributions become

$$p_{n,h}^d(\mathbf{x}_m, t) = \frac{\Gamma\left(\frac{n+1}{2}(d-h) + \frac{h}{2}\right)}{\Gamma\left(\frac{n+1}{2}(d-h) + \frac{h-m}{2}\right)} \frac{(c^2 t^2 - \|\mathbf{x}_m\|)^{\frac{n+1}{2}(d-h) - \frac{m-h}{2} - 1}}{\pi^{\frac{m}{2}} (ct)^{(n+1)(d-h)+h-2}},$$

(see formulas (2.26) and (2.27) of De Gregorio and Orsingher, 2012). By means of the same considerations used for the density function  $p_{n,h}(\underline{\mathbf{x}}_d, t)$  we obtain that  $p_{n,h}^d(\underline{\mathbf{x}}_m, t)$  is still solution of the EPD equation (4.19).

In the same spirit of the previous considerations, the function

$$\bar{f}_\beta(\underline{\mathbf{x}}_d, t) := \left( c^2 t^2 - \frac{R^4}{\|\underline{\mathbf{x}}_d\|^2} \right)^\beta, \quad \frac{R^2}{ct} < \|\underline{\mathbf{x}}_d\| \leq R, \beta \in \mathbb{R},$$

solves the following EPD partial differential equation with time and space varying coefficients

$$(4.21) \quad \frac{\partial^2 u}{\partial t^2} - \frac{a_\beta(\underline{\mathbf{x}}_d)}{t} \frac{\partial u}{\partial t} = c^2 \Delta u,$$

where

$$a_\beta(\underline{\mathbf{x}}_d) := 2\beta - 1 + \frac{R^4}{\|\underline{\mathbf{x}}_d\|^2} \left[ \frac{4-d}{\|\underline{\mathbf{x}}_d\|^2} + 2(\beta-1) \left( \frac{1-R^4/\|\underline{\mathbf{x}}_d\|^4}{c^2 t^2 - R^4/\|\underline{\mathbf{x}}_d\|^2} \right) \right].$$

For  $d=4$  and  $\beta=1$ , we have that the function  $\bar{f}_1(\underline{\mathbf{x}}_4, t)$  satisfies the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{t} \frac{\partial u}{\partial t} = c^2 \Delta u.$$

By setting

$$\bar{p}_{n,h}(\underline{\mathbf{x}}_d, t) := \frac{\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{n}{2}(d-h))} \frac{R^{2d}}{\pi^{\frac{d}{2}}(ct)^{(n+1)(d-h)+h-2}} \left( c^2 t^2 - \frac{R^4}{\|\underline{\mathbf{x}}_d\|^2} \right)^{\frac{n}{2}(d-h)-1},$$

with  $\frac{R^2}{ct} < \|\underline{\mathbf{x}}_d\| \leq R$ , we have that

$$\bar{f}_{\frac{n}{2}(d-h)-1}(\underline{\mathbf{x}}_d, t) = \frac{\pi^{\frac{d}{2}}(ct)^{(n+1)(d-h)+h-2}\Gamma(\frac{n}{2}(d-h))}{\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})} \frac{\|\underline{\mathbf{x}}_d\|^{2d}}{R^{2d}} \bar{p}_{n,h}(\underline{\mathbf{x}}_d, t).$$

Therefore, by exploiting the equation (4.21), we conclude that  $\bar{p}_{n,h}(\underline{\mathbf{x}}_d, t)$  is a solution of the following partial differential equation

$$(4.22) \quad \begin{aligned} & \frac{\partial^2 u}{\partial t^2} + \frac{2(2\beta+d) - a_\beta(\underline{\mathbf{x}}_d)}{t} \frac{\partial u}{\partial t} + \left[ \frac{(2\beta+d)((2\beta+d-1) - a_\beta(\underline{\mathbf{x}}_d))}{t^2} \right] u \\ & = c^2 \left[ \Delta + \frac{2d(3d-2)}{\|\underline{\mathbf{x}}_d\|^2} + \frac{4d\langle \underline{\mathbf{x}}_d, \nabla \rangle}{\|\underline{\mathbf{x}}_d\|^2} \right] u, \end{aligned}$$

where  $\beta = \frac{n}{2}(d-h) - 1$  and  $\nabla u := \text{grad} u$ . The equation (4.22) no longer has the structure of the EPD equation.

**4.4. On the unconditional probability distributions.** In order to obtain unconditional densities for  $\{\underline{\mathbf{x}}_d^*(t), t > 0\}$ , we should specify the probability distribution of  $\mathcal{N}(t)$ . Different choices of the above probability law lead to different unconditional densities of the reflecting random flight. We assume that the number of deviations  $\mathcal{N}(t) = \mathcal{N}_d(t)$ ,  $d \geq 2$ , at time  $t > 0$ , possesses the distribution of a weighted Poisson random variable. Let  $\{N(t), t > 0\}$  be a homogeneous Poisson process with rate  $\lambda > 0$ , a random variable  $\mathcal{N}_d(t)$  has weighted Poisson probability distribution if

$$(4.23) \quad P\{\mathcal{N}_d(t) = n\} = \frac{w_n P\{N(t) = n\}}{\sum_{k=0}^{\infty} w_k P\{N(t) = k\}}, \quad n \in \mathbb{N}_0,$$

where  $w_k$ s are non-negative weight functions with  $0 < \sum_{k=0}^{\infty} w_k P\{N(t) = k\} < \infty$  (for more details see Balakrishnan and Kozubowski, 2008). In our context, we choose the weights as follows

$$w_k = \frac{k!}{\Gamma((\frac{d-h}{2})k + \frac{d}{2})}$$

and then for this choice we obtain that

$$(4.24) \quad P\{\mathcal{N}_d(t) = n\} = \frac{1}{E_{\frac{d-h}{2}, \frac{d}{2}}(\lambda t)} \frac{(\lambda t)^n}{\Gamma((\frac{d-h}{2})n + \frac{d}{2})}, \quad n \in \mathbb{N}_0,$$

where  $E_{\alpha, \beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}$ ,  $x \in \mathbb{R}$ ,  $\alpha, \beta > 0$ , is the two-parameter Mittag-Leffler function. The random variable  $\mathcal{N}_d(t)$  with probability distribution (4.24) coincides with (4.1) in Beghin and Orsingher (2009).

**Theorem 2.** For  $t > \frac{R}{c}$  and by assuming (4.24), we have that the unconditional probability distribution of  $\{\mathbf{X}_d^*(t), t > 0\}$  becomes

$$(4.25) \quad P\{\mathbf{X}_d^*(t) \in d\mathbf{x}_d\} = p_h^*(\mathbf{x}_d, t) \prod_{k=1}^d dx_k + \frac{1}{E_{\frac{d-h}{2}, \frac{d}{2}}(\lambda t) \Gamma(\frac{d}{2})} \mu^*(d\mathbf{x}_d)$$

where

$$(4.26) \quad p_h^*(\mathbf{x}_d, t) = \frac{1}{(ct)^{d\pi^{\frac{d}{2}}}} \frac{1}{E_{\frac{d-h}{2}, \frac{d}{2}}(\lambda t)} \left\{ [\gamma_h(\|\mathbf{x}_d\|, t)]^{1-\frac{2}{d-h}} E_{\frac{d-h}{2}, \frac{d-h}{2}}(\gamma_h(\|\mathbf{x}_d\|, t)) \mathbf{1}_{B_R^d}(\mathbf{x}_d) \right. \\ \left. + \frac{R^{2d}}{\|\mathbf{x}_d\|^{2d}} \left[ \gamma_h\left(\frac{R^2}{\|\mathbf{x}_d\|}, t\right) \right]^{1-\frac{2}{d-h}} E_{\frac{d-h}{2}, \frac{d-h}{2}}\left(\gamma_h\left(\frac{R^2}{\|\mathbf{x}_d\|}, t\right)\right) \mathbf{1}_{C_{\frac{R^2}{ct}, R}^d}(\mathbf{x}_d) \right\},$$

with

$$\gamma_h(\|\mathbf{x}_d\|, t) := \lambda t \left( 1 - \frac{\|\mathbf{x}_d\|^2}{c^2 t^2} \right)^{\frac{d-h}{2}},$$

while  $\mu^*$  is the uniform distribution on  $\mathbb{S}_{\frac{R^2}{ct}}^{d-1}$ .

*Proof.* The second term in (4.25) arises from the following considerations. The probability distribution of the random flight  $\{\mathbf{X}_d^*(t), t > 0\}$  admits a singular component emerging in the case  $\mathcal{N}_d(t) = 0$ , that is if the initial direction of the motion does not change up to time  $t$ . For  $t > \frac{R}{c}$ , the circular inversion of  $\mathbb{S}_{ct}^{d-1}$  with respect to  $\mathbb{S}_R^{d-1}$  leads to  $\mathbb{S}_{\frac{R^2}{ct}}^{d-1}$ . Therefore, if there are no changes of direction in  $(0, t]$ , the reflected path reaches  $\mathbb{S}_{\frac{R^2}{ct}}^{d-1}$  and

$$P\left\{\mathbf{X}_d^*(t) \in \mathbb{S}_{\frac{R^2}{ct}}^{d-1}\right\} = P\{\mathcal{N}_d(t) = 0\} = \frac{1}{E_{\frac{d-h}{2}, \frac{d}{2}}(\lambda t) \Gamma(\frac{d}{2})}.$$

For the remaining part of the distribution  $P\{\mathbf{X}_d^*(t) \in d\mathbf{x}_d\}$  we can observe that

$$p_h^*(\mathbf{x}_d, t) = \sum_{n=1}^{\infty} p_{n,h}^*(\mathbf{x}_d, t) P\{\mathcal{N}_d(t) = n\}$$

where  $p_{n,h}(\mathbf{x}_d, t)$  is defined by (4.10). The same steps performed in the proof of Theorem 5 in De Gregorio and Orsingher (2012), lead to the result (4.26).  $\square$

**Remark 4.2.** For  $\frac{d-h}{2} = 1$ , that is  $d = 3$  for  $h = 1$  and  $d = 4$  for  $h = 2$ , the function (5.9) reduces to

$$p_h^*(\underline{\mathbf{x}}_d, t) = \frac{\lambda}{c^3 t^2 \pi^{\frac{d}{2}}} \frac{1}{E_{1, \frac{d}{2}}(\lambda t)} \left\{ \exp \left\{ \lambda t \left( 1 - \frac{\|\underline{\mathbf{x}}_d\|^2}{c^2 t^2} \right) \right\} \mathbf{1}_{B_R^d}(\underline{\mathbf{x}}_d) \right. \\ \left. + \frac{R^{2d}}{\|\underline{\mathbf{x}}_d\|^{2d}} \exp \left\{ \lambda t \left( 1 - \frac{R^4}{c^2 t^2 \|\underline{\mathbf{x}}_d\|^2} \right) \right\} \mathbf{1}_{C_{\frac{R^2}{ct}, R}^d}(\underline{\mathbf{x}}_d) \right\},$$

since  $E_{1,1}(x) = e^x$ .

We observe that the Dirichlet distribution (3.6) with  $h = 1$  and  $d = 2$  or  $h = 2$  and  $d = 4$ , reduces to the uniform law in  $S_n$ . Therefore, alternatively to (4.24), we can assume that

$$g(\mathcal{I}_n; t) = \frac{n!}{t^n} \mathbf{1}_{S_n}(\mathcal{I}_n).$$

In other words, instead of  $\mathcal{N}_d(t)$ , we can suppose that the changes of direction are governed by a homogeneous Poisson process  $\{N(t), t > 0\}$  with rate  $\lambda > 0$ . Under these assumptions, the unconditional distributions of random flights  $\{\underline{\mathbf{x}}_d(t), t > 0\}$ ,  $d = 2, 4$ , are given by

$$(4.27) \quad p_h(\underline{\mathbf{x}}_d, t) = \sum_{n=1}^{\infty} p_{n,h}(\underline{\mathbf{x}}_d, t) P\{N(t) = n\} \\ = \begin{cases} \frac{\lambda e^{-\lambda t}}{2\pi c} \frac{e^{\frac{\lambda}{c} \sqrt{c^2 t^2 - \|\underline{\mathbf{x}}_2\|^2}}}{\sqrt{c^2 t^2 - \|\underline{\mathbf{x}}_2\|^2}} \mathbf{1}_{B_{ct}^2}(\underline{\mathbf{x}}_2), & d = 2, h = 1, \\ \frac{\lambda}{c^4 t^3 \pi^2} e^{-\frac{\lambda}{c^2 t} \|\underline{\mathbf{x}}_4\|^2} \left\{ 2 + \frac{\lambda}{c^2 t} (c^2 t^2 - \|\underline{\mathbf{x}}_4\|^2) \right\} \mathbf{1}_{B_{ct}^4}(\underline{\mathbf{x}}_4), & d = 4, h = 2. \end{cases}$$

(for the case  $d = 2, h = 1$ , see (1.2) of Stadje, 1987, (18) of Masoliver *et al.*, 1993, (20) of Kolesnik and Orsingher, 2005, and for the case  $d = 4, h = 2$ , see formula (3.2) of Orsingher and De Gregorio, 2007).

Therefore, if we assume that the changes of direction are governed by the homogeneous Poisson process  $\{N(t), t > 0\}$ , the related reflecting random flights  $\{\underline{\mathbf{x}}_d^*(t), t > 0\}$ ,  $d = 2, 4$ , have unconditional density functions (in a generalized sense) equal to

$$(4.28) \quad f_h^*(\underline{\mathbf{x}}_d, t) = p_h(\underline{\mathbf{x}}_d, t) \mathbf{1}_{B_R^d}(\underline{\mathbf{x}}_d) + \frac{R^{2d}}{\|\underline{\mathbf{x}}_d\|^{2d}} p_h \left( R^2 \frac{\underline{\mathbf{x}}_d}{\|\underline{\mathbf{x}}_d\|^2}, t \right) \mathbf{1}_{C_{\frac{R^2}{ct}, R}^d}(\underline{\mathbf{x}}_d) \\ + \frac{e^{-\lambda t}}{\text{area}(\mathbb{S}_{R^2/ct}^{d-1})} \delta_{\{R^2/ct\}}(\|\underline{\mathbf{x}}_d\|),$$

where  $p_h(\underline{\mathbf{x}}_d, t)$  is given by (4.27) and the  $\frac{e^{-\lambda t}}{\text{area}(\mathbb{S}_{R^2/ct}^{d-1})} \delta_{\{R^2/ct\}}(\|\underline{\mathbf{x}}_d\|)$  emerges if  $N(t) = 0$ .

**Theorem 3.** For  $t > \frac{R}{c}$ ,  $0 < r \leq R$  and  $d = 2, h = 1$ , we obtain that

$$(4.29) \quad P\{D_2^*(t) < r\} = \left[ 1 - \exp \left\{ -\lambda t + \frac{\lambda}{c} \sqrt{c^2 t^2 - r^2} \right\} \right] \mathbf{1}_{(0, R]}(r) \\ + \exp \left\{ -\lambda t + \frac{\lambda}{c} \sqrt{c^2 t^2 - R^4/r^2} \right\} \mathbf{1}_{(R^2/ct, R]}(r)$$

*Proof.* Let us fix  $r \in (0, R]$  and  $d = 2, h = 1$ . We have that

$$P\{D_2^*(t) < r\} = P\{\underline{\mathbf{x}}_2^*(t) \in B_r^2\} \\ = \int_{B_r^2} p_1(\underline{\mathbf{x}}_2, t) dx_1 dx_2 + \int_{C_{\frac{R^2}{ct}, R}^2 \cap B_r^2} \frac{R^4}{\|\underline{\mathbf{x}}_2\|^4} p_1 \left( \frac{R^2}{\|\underline{\mathbf{x}}_2\|^2}, t \right) dx_1 dx_2.$$

where  $p_1(\underline{\mathbf{x}}_2, t)$  is given by (4.27). If  $r < R^2/ct$ , it is clear that  $C_{\frac{R^2}{ct}, R}^2 \cap B_r^2 = \emptyset$  and then

$$\begin{aligned} P\{\underline{\mathbf{X}}_2^*(t) \in B_r^2\} &= \int_{B_r^2} \frac{\lambda e^{-\lambda t}}{2\pi c} \frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2 - \|\underline{\mathbf{x}}_2\|^2}}}{\sqrt{c^2t^2 - \|\underline{\mathbf{x}}_2\|^2}} dx_1 dx_2 \\ &= \int_0^r \frac{\lambda e^{-\lambda t}}{c} \rho \frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2 - \rho^2}}}{\sqrt{c^2t^2 - \rho^2}} d\rho \\ &= 1 - \exp\left\{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - r^2}\right\} \end{aligned}$$

For  $r \in (R^2/ct, R]$ , we have that  $C_{\frac{R^2}{ct}, R}^2 \cap B_r^2 = C_{\frac{R^2}{ct}, r}^2$ . Therefore

$$\begin{aligned} \int_{C_{\frac{R^2}{ct}, r}^2} \frac{\lambda e^{-\lambda t}}{2\pi c} \frac{R^4}{\|\underline{\mathbf{x}}_2\|^4} \frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2 - R^4/\|\underline{\mathbf{x}}_2\|^2}}}{\sqrt{c^2t^2 - R^4/\|\underline{\mathbf{x}}_2\|^2}} dx_1 dx_2 &= \int_{R^2/ct}^r \frac{\lambda e^{-\lambda t}}{c} \frac{R^4}{\rho^3} \frac{e^{\frac{\lambda}{c}\sqrt{c^2t^2 - R^4/\rho^2}}}{\sqrt{c^2t^2 - R^4/\rho^2}} d\rho \\ &= \exp\left\{-\lambda t + \frac{\lambda}{c}\sqrt{c^2t^2 - R^4/r^2}\right\} - \exp\{-\lambda t\}. \end{aligned}$$

If  $R^2/ct < r \leq R$ , we also have to consider the discrete part of the distribution of  $\{\underline{\mathbf{X}}_2^*(t), t > 0\}$ . Therefore

$$P\{D_2^*(t) = R^2/ct\} = P\{\underline{\mathbf{X}}_2^*(t) \in \mathbb{S}_{R^2/ct}^{d-1}\} = P\{N(t) = 0\} = e^{-\lambda t}.$$

This last fact concludes the proof of the theorem.  $\square$

## 5. REFLECTING RANDOM FLIGHTS ON HYPERPLANES

**5.1. Definitions and probability distributions.** In this section we introduce a random flight bouncing off a hyperplane. Let  $H(\underline{\mathbf{a}}_d, b) := \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \langle \underline{\mathbf{a}}_d, \underline{\mathbf{x}}_d \rangle = b; \underline{\mathbf{a}}_d \in \mathbb{R}^d, b \in \mathbb{R}\}$  be a hyperplane in  $\mathbb{R}^d$ . A random flight starting from the origin of  $\mathbb{R}^d$ , for sufficiently large values of  $t$  can be located beyond the hyperplane  $H(\underline{\mathbf{a}}_d, b)$ . The spherical set of the possible positions  $B_{ct}^d$  is therefore composed by the set  $L_{ct}^d := L_{ct}^d(\underline{\mathbf{a}}_d, b) := \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \|\underline{\mathbf{x}}_d\|^2 < c^2t^2, \langle \underline{\mathbf{a}}_d, \underline{\mathbf{x}}_d \rangle < b\}$  pertaining to the sample paths which have not crossed  $H(\underline{\mathbf{a}}_d, b)$  and the set  $U_{ct}^d := U_{ct}^d(\underline{\mathbf{a}}_d, b) := \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \|\underline{\mathbf{x}}_d\|^2 < c^2t^2, \langle \underline{\mathbf{a}}_d, \underline{\mathbf{x}}_d \rangle \geq b\}$  related to the trajectories which have gone beyond the hyperplane. Of course, if no deviation is recorded by the random flight up to time  $t$ , the moving particle attains the sphere  $\mathbb{S}_{ct}^{d-1}$ , which therefore can be split as  $\mathbb{S}_{ct}^{d-1} = \partial L_{ct}^d \cup \partial U_{ct}^d$ , where  $\partial L_{ct}^d := \partial L_{ct}^d(\underline{\mathbf{a}}_d, b) := \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \|\underline{\mathbf{x}}_d\|^2 = c^2t^2, \langle \underline{\mathbf{a}}_d, \underline{\mathbf{x}}_d \rangle < b\}$  and  $\partial U_{ct}^d := \partial U_{ct}^d(\underline{\mathbf{a}}_d, b) := \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \|\underline{\mathbf{x}}_d\|^2 = c^2t^2, \langle \underline{\mathbf{a}}_d, \underline{\mathbf{x}}_d \rangle \geq b\}$ .

The reflection of the sample paths crossing  $H(\underline{\mathbf{a}}_d, b)$  is described in detail in Appendix B. Substantially, the incoming and reflected sample paths form the same angle  $\theta$  w.r.t. the normal to the hyperplane. Let  $\nu : \mathbb{R}^d \rightarrow \mathbb{R}$  be the reflecting (bijective) operator with respect to the hyperplane  $H(\underline{\mathbf{a}}_d, b)$  defined as

$$\nu(\underline{\mathbf{x}}_d) := \underline{\mathbf{x}}_d + 2 \frac{b - \langle \underline{\mathbf{a}}_d, \underline{\mathbf{x}}_d \rangle}{\langle \underline{\mathbf{a}}_d, \underline{\mathbf{a}}_d \rangle} \underline{\mathbf{a}}_d.$$

Now, we are able to define the reflecting random flight. Let  $t' := \inf\{t : H(\underline{\mathbf{a}}_d, b) \cap B_{ct}^d \neq \emptyset\}$ .

**Definition 3.** The reflecting random flight  $\{\underline{\mathbf{X}}_d'(t), t > 0\}$ , reflected by the hyperplane  $H(\underline{\mathbf{a}}_d, b)$  is constructed by means of the free process  $\{\underline{\mathbf{X}}_d(t), t > 0\}$  as follows: 1) if  $t < t'$ , then  $\underline{\mathbf{X}}_d'(t) = \underline{\mathbf{X}}_d(t)$ ; 2) if  $t \geq t'$  and at least one change of direction happens during the time interval  $[0, t]$ , we have that

$$(5.1) \quad \underline{\mathbf{X}}_d'(t) = \underline{\mathbf{X}}_d(t) \mathbf{1}_{L_{ct}^d}(\underline{\mathbf{X}}_d(t)) + \nu(\underline{\mathbf{X}}_d(t)) \mathbf{1}_{U_{ct}^d}(\underline{\mathbf{X}}_d(t)),$$

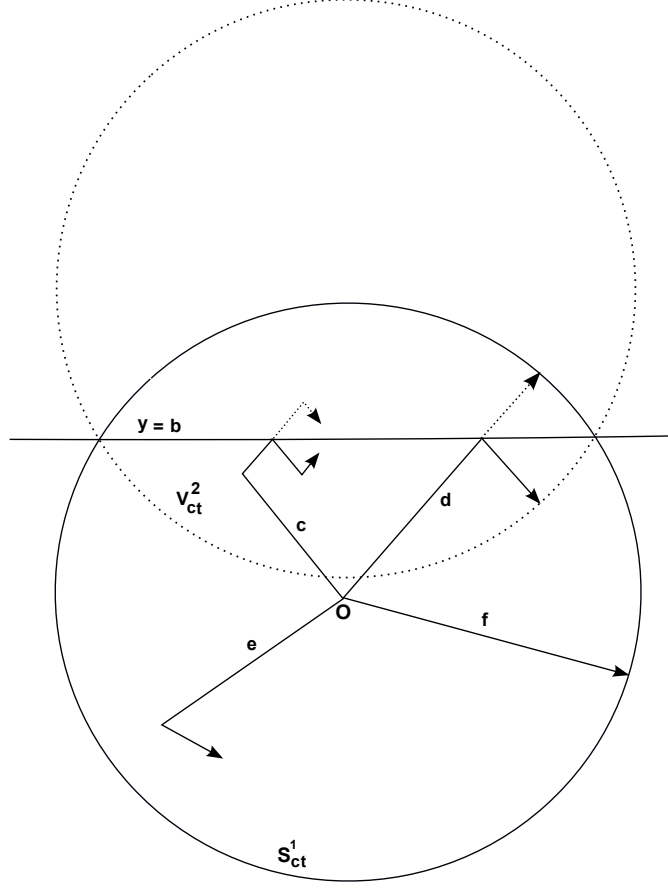


FIGURE 2. Four typical sample paths are depicted. The trajectories **c** and **d** are reflected in the set  $V_{ct}^d$ , while **e** and **f** never cross the reflecting surface  $y = b$ .

while if no deviation up to time  $t$  is recorded

$$(5.2) \quad \underline{\mathbf{X}}'_d(t) = \underline{\mathbf{X}}_d(t) \mathbf{1}_{\partial L_{ct}^d}(\underline{\mathbf{X}}_d(t)) + \nu(\underline{\mathbf{X}}_d(t)) \mathbf{1}_{\partial U_{ct}^d}(\underline{\mathbf{X}}_d(t)).$$

The set of points of  $B_{ct}^d$  obtained by reflection  $\nu$  around  $H(\underline{a}_d, b)$  is denoted by  $V_{ct}^d$ , that is  $V_{ct}^d := V_{ct}^d(\underline{a}_d, b) := \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \|\nu(\underline{\mathbf{x}}_d)\|^2 < c^2 t^2, \langle \underline{a}_d, \underline{\mathbf{x}}_d \rangle \leq b\}$ , while  $\partial V_{ct}^d$  stands for the set obtained by the reflection of  $\partial U_{ct}^d$  (see Figure 2). The reflection at the hyperplane preserves the form of the sample paths of the free random flight; i.e. the sample paths of  $\{\underline{\mathbf{X}}'_d(t), t > 0\}$  are straight lines as well as the trajectories of the free random flights (see Figure 2). Furthermore, the property Q4) guarantees that the reflected trajectories are symmetrically specular w.r.t. the hyperplane  $H(\underline{a}_d, b)$ .

The conditional distributions of the reflecting random flight  $\{\underline{\mathbf{X}}'_d(t), t > 0\}$  are given in the next theorem.

**Theorem 4.** *If  $\mathcal{N}(t) = n$ , with  $n \geq 1$ , the process  $\{\underline{\mathbf{X}}'_d(t), t > 0\}$ , has the following conditional density functions*

$$(5.3) \quad p'_n(\underline{\mathbf{x}}_d, t) = \begin{cases} p_n(\underline{\mathbf{x}}_d, t) \mathbf{1}_{B_{ct}^d}(\underline{\mathbf{x}}_d), & t \leq t' \\ p_n(\underline{\mathbf{x}}_d, t) \mathbf{1}_{L_{ct}^d}(\underline{\mathbf{x}}_d) + p_n(\nu(\underline{\mathbf{x}}_d), t) \mathbf{1}_{V_{ct}^d}(\underline{\mathbf{x}}_d), & t > t'. \end{cases}$$



where  $p_n(\underline{\mathbf{x}}_d, t)$  is equal to (3.1).

*Proof.* The case  $t \leq t'$  is trivial. We assume that  $t > t'$ . Let  $A$  be a Borel set such that  $A \cap H(\underline{a}_d, b) \neq \emptyset$ . We observe that

$$(5.4) \quad \begin{aligned} P_n\{\underline{\mathbf{X}}'_d(t) \in A\} &= P_n\{\underline{\mathbf{X}}_d(t) \in A \cap L_{ct}^d\} + P_n\{\nu(\underline{\mathbf{X}}_d(t)) \in A \cap V_{ct}^d\} \\ &= \int_{A \cap L_{ct}^d} p_n(\underline{\mathbf{x}}_d, t) \prod_{k=1}^d dx_k + P_n\{\nu(\underline{\mathbf{X}}_d(t)) \in A \cap V_{ct}^d\}. \end{aligned}$$

Let now

$$B_A := \left\{ \underline{\mathbf{y}}_d \in \mathbb{R}^d : \underline{\mathbf{x}}_d = \nu(\underline{\mathbf{y}}_d) \in A \cap V_{ct}^d \right\}$$

and thus, by means of Jacobi's transformation formula, we have that

$$(5.5) \quad \begin{aligned} P_n\{\nu(\underline{\mathbf{X}}_d(t)) \in A \cap V_{ct}^d\} &= P_n\{\underline{\mathbf{X}}_d(t) \in B_A\} \\ &= \int_{B_A} p_n(\underline{\mathbf{y}}_d, t) \prod_{k=1}^d dy_k \\ &= \int_{A \cap V_{ct}^d} p_n(\nu^{-1}(\underline{\mathbf{x}}_d), t) |\det(J_{\nu^{-1}}(\underline{\mathbf{x}}_d))| \prod_{k=1}^d dx_k \\ &= \int_{A \cap V_{ct}^d} p_n(\nu(\underline{\mathbf{x}}_d), t) \prod_{k=1}^d dx_k \end{aligned}$$

where in the last step we have exploited the facts:  $\nu = \nu^{-1}$  (which follows from the property Q3)) and  $|\det(J_{\nu}(\underline{\mathbf{x}}_d))| = 1$ . From (5.4) and (5.5) the result (5.3) immediately follows.  $\square$

**Remark 5.1.** *In view of the property (B.3), the reflecting random flights introduced by Definition 3 are no longer isotropic. Indeed, for  $t > t'$ , in the density function (5.3) appears*

$$p_n(\nu(\underline{\mathbf{x}}_d), t) = p_n(\|\nu(\underline{\mathbf{x}}_d)\|, t)$$

*which does not depend only on the Euclidean distance  $\|\underline{\mathbf{x}}_d\|$ .*

Now, we consider reflecting Dirichlet random flights. The assumption (3.6) implies that the reflecting process  $\{\underline{\mathbf{X}}'_d(t), t > 0\}$  has probability law (5.3) given by

$$(5.6) \quad \begin{aligned} p'_{n,h}(\underline{\mathbf{x}}_d, t) &= \frac{\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{n}{2}(d-h))} \frac{1}{\pi^{\frac{d}{2}}(ct)^{(n+1)(d-h)+h-2}} \\ &\quad \times \left[ (c^2 t^2 - \|\underline{\mathbf{x}}_d\|^2)^{\frac{n}{2}(d-h)-1} \mathbf{1}_{L_{ct}^d}(\underline{\mathbf{x}}_d) + (c^2 t^2 - \|\nu(\underline{\mathbf{x}}_d)\|^2)^{\frac{n}{2}(d-h)-1} \mathbf{1}_{V_{ct}^d}(\underline{\mathbf{x}}_d) \right], \end{aligned}$$

where  $t > t'$  and  $\|\nu(\underline{\mathbf{x}}_d)\|^2$  is given by (B.3). In the special cases (i)-(iv) mentioned in Remark 4.1, the function (5.6) becomes

$$(5.7) \quad p'_{n,h}(\underline{\mathbf{x}}_d, t) = \frac{1}{\text{area}(\mathbb{S}_{ct}^{d-1})} \left[ \mathbf{1}_{L_{ct}^d}(\underline{\mathbf{x}}_d) + \mathbf{1}_{V_{ct}^d}(\underline{\mathbf{x}}_d) \right], \quad t > t'.$$

By assuming that the random number of changes of direction has probability law (4.24), for  $t > t'$ , we have that the unconditional probability distribution of  $\{\underline{\mathbf{X}}'_d(t), t > 0\}$  becomes

$$(5.8) \quad P\{\underline{\mathbf{X}}'_d(t) \in d\underline{\mathbf{x}}_d\} = p'_h(\underline{\mathbf{x}}_d, t) \prod_{k=1}^d dx_k + \frac{1}{E_{\frac{d-h}{2}, \frac{d}{2}}(\lambda t) \Gamma(\frac{d}{2})} \mu'(d\underline{\mathbf{x}}_d)$$

where

$$(5.9) \quad p'_h(\underline{\mathbf{x}}_d, t) = \frac{1}{(ct)^d \pi^{\frac{d}{2}}} \frac{1}{E_{\frac{d-h}{2}, \frac{d}{2}}(\lambda t)} \left\{ [\gamma_h(\|\underline{\mathbf{x}}_d\|, t)]^{1-\frac{2}{d-h}} E_{\frac{d-h}{2}, \frac{d-h}{2}}(\gamma_h(\|\underline{\mathbf{x}}_d\|, t)) \mathbf{1}_{L_{ct}^d}(\underline{\mathbf{x}}_d) \right. \\ \left. + [\gamma_h(\|\nu(\underline{\mathbf{x}}_d)\|, t)]^{1-\frac{2}{d-h}} E_{\frac{d-h}{2}, \frac{d-h}{2}}(\gamma_h(\|\nu(\underline{\mathbf{x}}_d)\|, t)) \mathbf{1}_{V_{ct}^d}(\underline{\mathbf{x}}_d) \right\},$$

with

$$\gamma_h(\|\underline{\mathbf{x}}_d\|, t) := \lambda t \left( 1 - \frac{\|\underline{\mathbf{x}}_d\|^2}{c^2 t^2} \right)^{\frac{d-h}{2}},$$

and  $\mu'$  represents the uniform law on  $\partial L_{ct}^d \cup \partial V_{ct}^d$ .

**Remark 5.2.** *It is not hard to prove that the function*

$$\hat{f}_\beta(\underline{\mathbf{x}}_d, t) := (c^2 t^2 - \|\nu(\underline{\mathbf{x}}_d)\|^2)^\beta, \quad \underline{\mathbf{x}}_d \in V_{ct}^d, \beta \in \mathbb{R},$$

*is a solution of the partial differential equation (4.19). Therefore, the second component appearing in (5.9), that is*

$$\hat{p}_{n,h}(\underline{\mathbf{x}}_d, t) := \frac{\Gamma(\frac{n+1}{2}(d-h) + \frac{h}{2})}{\Gamma(\frac{n}{2}(d-h))} \frac{(c^2 t^2 - \|\nu(\underline{\mathbf{x}}_d)\|^2)^{\frac{n}{2}(d-h)-1}}{\pi^{\frac{d}{2}}(ct)^{(n+1)(d-h)+h-2}}, \quad \underline{\mathbf{x}}_d \in V_{ct},$$

*is a solution of the EPD equation (4.20).*

**5.2. On the probability law of the distance from the origin.** For the sake of simplicity, we set  $\underline{e}_d = e_d := (0, \dots, 0, 1)$ . Therefore, the hyperplane becomes

$$H(e_d, b) = \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : x_d = b; b > 0\}$$

and the reflection map becomes

$$\nu(\underline{\mathbf{x}}_d) := \underline{\mathbf{x}}_d + 2(b - x_d),$$

with

$$\|\nu(\underline{\mathbf{x}}_d)\|^2 = \|\underline{\mathbf{x}}_d\|^2 + 4b^2 - 4bx_d.$$

Under the above assumption, we have that

$$L_{ct}^d = \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \|\underline{\mathbf{x}}_d\|^2 < c^2 t^2, x_d < b\}, \\ V_{ct}^d = \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \|\underline{\mathbf{x}}_d\|^2 + 4b^2 - 4bx_d < c^2 t^2, x_d \leq b\}.$$

Let  $\{D'_d(t), t > 0\}$  where  $D'_d(t) = \|\underline{\mathbf{x}}'_d(t)\|$ . Since the process  $\{\underline{\mathbf{x}}'_d(t), t > 0\}$  is not isotropic, the probability distribution of  $\{D'_d(t), t > 0\}$  is more complicated than (4.9). Now, we consider the distribution function

$$P_n\{D'_d(t) < r\}$$

with  $0 < r < ct$ . For  $t > t'$ , we distinguish the following three cases (see Figure 3):

1.  $0 < r < ct - 2b$ , where the ball  $B_r^d$  does not intersect  $V_{ct}^d$ ;
2.  $ct - 2b < r < b$ , where  $B_r^d$  intersects  $V_{ct}^d$  but does not overlap  $H(e_d, b)$ ;
3.  $b < r < ct$ , where  $B_r^d$  intersects  $V_{ct}^d$  and  $H(e_d, b)$ .

In the case (i), we simply have that

$$(5.10) \quad P_n\{D'_d(t) < r\} = P_n\{D_d(t) < r\} = P_n\{\underline{\mathbf{x}}_d(t) \in B_r^d\}.$$

In the second case, we must take into account that in  $V_{ct}^d \cap B_r^d$  we meet reflected sample paths and thus

$$(5.11) \quad P_n\{D'_d(t) < r\} = P_n\{\underline{\mathbf{x}}_d(t) \in B_r^d\} + P_n\{\nu(\underline{\mathbf{x}}_d(t)) \in V_{ct}^d \cap B_r^d\}.$$

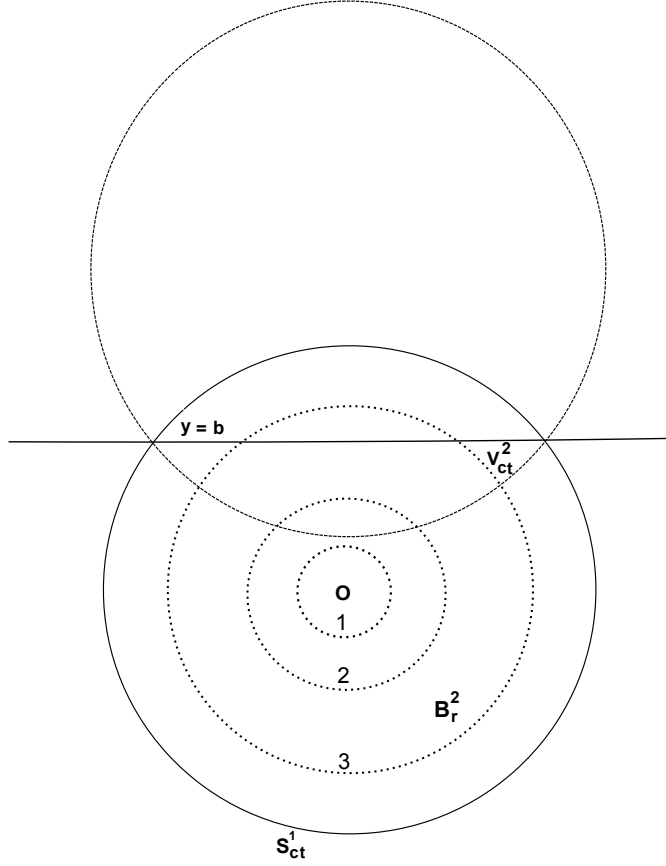


FIGURE 3. The picture represents cases 1., 2. and 3. emerging in the analysis of  $P_n\{D'_d(t) < r\}$ .

In the third case  $B_r^d = L_r^d \cup U_r^d$  and thus

$$(5.12) \quad P_n\{D'_d(t) < r\} = P_n\{\underline{\mathbf{X}}_d(t) \in L_r\} + P_n\{\nu(\underline{\mathbf{X}}_d(t)) \in V_{ct}^d \cap L_r\}.$$

The reader should consider that sample paths crossing  $x_d = b$  and outside  $U_r^d$  can contribute to probability (5.12) because the reflected trajectories lie within  $V_{ct}^d \cap L_r$ . All these considerations can be summarized as follows

$$(5.13) \quad P_n\{D'_d(t) < r\} = \begin{cases} \int_{B_r^d} p_n(\underline{\mathbf{x}}_d, t) \prod_{k=1}^d dx_k, & 0 < r < ct - 2b, \\ \int_{B_r^d} p_n(\underline{\mathbf{x}}_d, t) \prod_{k=1}^d dx_k + \int_{V_{ct}^d \cap B_r^d} p_n(\nu(\underline{\mathbf{x}}_d), t) \prod_{k=1}^d dx_k, & ct - 2b < r < b, \\ \int_{L_r^d} p_n(\underline{\mathbf{x}}_d, t) \prod_{k=1}^d dx_k + \int_{V_{ct}^d \cap L_r^d} p_n(\nu(\underline{\mathbf{x}}_d), t) \prod_{k=1}^d dx_k, & b < r < ct. \end{cases}$$

#### APPENDIX A. REFLECTION IN SPHERES

We recall the basic facts about the circular inversion or reflection of a point inside a sphere (see, for instance, Ratcliffe, 2006, and Wong, 2009). If we consider a point  $\underline{\mathbf{x}}_d$  inside  $\mathbb{S}_R^{d-1}(\underline{\mathbf{x}}_d^0)$ , having polar coordinates equal to  $(r, \underline{\theta}_{d-1})$ , we can find another point  $\underline{\mathbf{x}}'_d$  in the space  $\mathbb{R}^d$  with

polar coordinates given by  $(r', \underline{\theta}_{d-1})$  ( $R < r'$  and the same angle  $\underline{\theta}_{d-1}$ ), such that

$$rr' = R^2.$$

or equivalently

$$\|\underline{\mathbf{x}}_d - \underline{\mathbf{x}}_d^0\| \cdot \|\underline{\mathbf{x}}'_d - \underline{\mathbf{x}}_d^0\| = R^2.$$

The point  $\underline{\mathbf{x}}_d$  is called the inverse point of  $\underline{\mathbf{x}}'_d$  with respect to  $\mathbb{S}_R^{d-1}(\underline{\mathbf{x}}_d^0)$ . The circular inversion in  $\mathbb{S}_R^{d-1}(\underline{\mathbf{x}}_d^0)$  is defined as the bijective map  $\mu_{R, \underline{\mathbf{x}}_d^0} : \mathbb{R}^d \setminus \{O\} \rightarrow \mathbb{R}^d \setminus \{O\}$  defined as follows

$$(A.1) \quad \mu_{R, \underline{\mathbf{x}}_d^0}(\underline{\mathbf{x}}_d) = R^2 \frac{\underline{\mathbf{x}}_d - \underline{\mathbf{x}}_d^0}{\|\underline{\mathbf{x}}_d - \underline{\mathbf{x}}_d^0\|^2} + \underline{\mathbf{x}}_d^0.$$

We set  $\mu_{R, O}(\underline{\mathbf{x}}_d) := \mu_R(\underline{\mathbf{x}}_d) = R^2 \frac{\underline{\mathbf{x}}_d}{\|\underline{\mathbf{x}}_d\|^2}$ . The map  $\mu_R$  has the following properties:

- P1) the points inside the sphere are taken to points outside it and vice versa;
- P2)  $\mu_R(\underline{\mathbf{x}}_d) = \underline{\mathbf{x}}_d$  if and only if  $\underline{\mathbf{x}}_d \in \mathbb{S}_R^{d-1}$ ;
- P3)  $(\mu_R \circ \mu_R)(\underline{\mathbf{x}}_d) = \underline{\mathbf{x}}_d$  for all  $\mathbb{R}^d \setminus \{O\}$ ;
- P4) the map  $\mu_R$  is conformal and reverses orientation (that is  $\det(\mu'_R(\underline{\mathbf{x}}_d)) < 0$ );
- P5) the inversion  $\mu_R$  maps straight lines into a straight line or sphere. In other words, lines passing through the center of inversion are mapped into themselves; while lines not passing through the center of inversion are mapped into spheres passing through the center.

## APPENDIX B. REFLECTION IN HYPERPLANES

For the main aspects on the reflection in hyperplanes consult Ratcliffe (2006). Let us consider a hyperplane of  $\mathbb{R}^d$  given by

$$(B.1) \quad H(\underline{a}_d, b) = \{\underline{\mathbf{x}}_d \in \mathbb{R}^d : \langle \underline{a}_d, \underline{\mathbf{x}}_d \rangle = b; \underline{a}_d \in \mathbb{R}^d, b \in \mathbb{R}\}.$$

Let  $\nu : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the reflection map in the hyperplane  $H(\underline{a}_d, b)$  which is a bijection defined as

$$(B.2) \quad \nu(\underline{\mathbf{x}}_d) := \underline{\mathbf{x}}_d + 2 \frac{b - \langle \underline{a}_d, \underline{\mathbf{x}}_d \rangle}{\langle \underline{a}_d, \underline{a}_d \rangle} \underline{a}_d,$$

with

$$(B.3) \quad \|\nu(\underline{\mathbf{x}}_d)\|^2 = \|\underline{\mathbf{x}}_d\|^2 + \frac{4b^2 - 4b\langle \underline{a}_d, \underline{\mathbf{x}}_d \rangle}{\langle \underline{a}_d, \underline{a}_d \rangle}$$

The map  $\nu(\underline{\mathbf{x}}_d)$  is defined as the mirror image of  $\underline{\mathbf{x}}_d$  across  $H(\underline{a}_d, b)$ . Furthermore,  $\nu$  has the following properties (see, for instance, Ratcliffe, 2006):

- Q1) the points inside the hyperplane are taken to points outside it and vice versa;
- Q2)  $\nu(\underline{\mathbf{x}}_d) = \underline{\mathbf{x}}_d$  if and only if  $\underline{\mathbf{x}}_d \in H(\underline{a}_d, b)$ ;
- Q3)  $(\nu \circ \nu)(\underline{\mathbf{x}}_d) = \underline{\mathbf{x}}_d$  for all  $\underline{\mathbf{x}}_d \in \mathbb{R}^d$ ;
- Q4) the map  $\nu$  is conformal and reverses orientation (that is  $\det(\nu'(\underline{\mathbf{x}}_d)) < 0$ );
- Q5)  $\nu$  is an isometry.

## REFERENCES

- [1] Aryasova, O., De Gregorio, A., Orsingher, E. (2013) Reflecting diffusions and hyperbolic Brownian motions in multidimensional spheres, *Lithuanian Mathematical Journal*, **53**, 241-263.
- [2] Balakrishnan, N., Kozubowski, T. (2008) A class of weighted Poisson processes, *Statistics and Probability Letters*, **78**, 2346-2352.
- [3] Beghin, L., Orsingher, E. (2009) Fractional Poisson processes and related planar random motions, *Electronic Journal of Probability*, **14**, 1790-1826.

- [4] Beghin, L., Orsingher, E. (2010) Moving randomly amid scattered obstacles, *Stochastics*, **82**, 201-229.
- [5] De Gregorio, A. (2012) On random flights with non-uniformly distributed directions, *Journal of Statistical Physics*, **147**, 382-411.
- [6] De Gregorio, A. (2014) A family of random walks with generalized Dirichlet steps, *Journal of Mathematical Physics*, **55**, 023302.
- [7] De Gregorio, A., Orsingher, E. (2012) Flying randomly in  $\mathbb{R}^d$  with Dirichlet displacements, *Stochastic Processes and their Applications*, **122**, 676-713.
- [8] Franceschetti, M. (2007) When a random walk of fixed length can lead uniformly anywhere inside a hypersphere, *Journal of Statistical Physics*, **127**, 813-823.
- [9] Garra, R., Orsingher, E. (2014) Random flights governed by Klein-Gordon-type partial differential equations, *Stochastic Processes and their Applications*, **124**, 2171-2187.
- [10] Ghosh, A., Rastegar, R., Roitershtein, A. (2014) On a directionally reinforced random walk, *Proceedings of the American Mathematical Society*, **142**, 3269-3283.
- [11] Hughes, B. D. (1995) *Random walks and random environment. Vol. 1. Random walks*. Oxford Science Publications.
- [12] Kolesnik, A.D., Orsingher (2005) A planar random motion with an infinite number of directions controlled by the damped wave equation, *Journal of Applied Probability*, **42**, 1168-1182.
- [13] Le Caër, G. (2010) A Pearson random walk with steps of uniform orientation and Dirichlet distributed lengths, *Journal of Statistical Physics*, **140**, 728-751.
- [14] Le Caër, G. (2011) A new family of solvable Pearson-Dirichlet random walks, *Journal of Statistical Physics*, **144**, 23-45.
- [15] Letac, G., Piccioni, M. (2013) Dirichlet random walks, To appear in *Journal of Applied Probability*, <http://arxiv.org/abs/1310.6279>.
- [16] Martens, K., Angelani, L., Di Leonardo, R., Bocquet, L. (2012) Probability distributions for the run-and-tumble bacterial dynamics: an analogy to the Lorentz model, *The European Physical Journal E*, **35**: 84.
- [17] Masoliver, J., Porrà, J.M., Weiss, G.H. (1993) Some two and three-dimensional persistent random walks, *Physica A*, **193**, 469-482.
- [18] Orsingher, E., De Gregorio, A. (2007) Random flights in higher spaces, *Journal of Theoretical Probability*, **20**, 769-806.
- [19] Pogorui, A.A., Rodriguez-Dagnino, R.M. (2011) Isotropic random motion at finite speed with  $K$ -Erlang distributed direction alternations, *Journal of Statistical Physics*, **145**, 102-112.
- [20] Pogorui, A.A., Rodriguez-Dagnino, R.M. (2012) Random motion with uniformly distributed directions and random velocity, *Journal of Statistical Physics*, **147**, 1216-1225.
- [21] Pogorui, A.A., Rodriguez-Dagnino, R.M. (2013) Random motion with gamma steps in higher dimensions, *Statistics and Probability Letters*, **83**, 1638-1643.
- [22] Ratcliffe, J.G. (2006) *Foundations of Hyperbolic Manifolds*. Second edition. Graduate Texts in Mathematics, 149. Springer, New York.
- [23] Reimberg, P.H., Abramo, L.R. (2013) CMB and random flights: temperature and polarization in position space, *Journal of Cosmology and Astroparticle Physics*, 06(043).
- [24] Stadje W. (1987) The exact probability distribution of a two-dimensional random walk, *Journal of Statistical Physics*, **46**, 207-216.
- [25] Stadje, W. (1989) Exact probability distributions for noncorrelated random walk models, *Journal of Statistical Physics*, **56**, 415-435.
- [26] Wong, Y.L. (2009) *An introduction to Geometry*. [http://www.math.nus.edu.sg/~matwyl/Notes\\_MA2219.pdf](http://www.math.nus.edu.sg/~matwyl/Notes_MA2219.pdf).

DIPARTIMENTO DI SCIENZE STATISTICHE, "SAPIENZA" UNIVERSITY OF ROME, P.LE ALDO MORO, 5 - 00185, ROME, ITALY

E-mail address: [alessandro.degregorio@uniroma1.it](mailto:alessandro.degregorio@uniroma1.it)

E-mail address: [enzo.orsingher@uniroma1.it](mailto:enzo.orsingher@uniroma1.it)